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The plus-construction, Bousfield localization, and derived completion

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ABSTRACT

We define a plus-construction on connective augmented algebras over operads in symmetric spectra using Quillen homology. For associative and commutative algebras, we show that this plus-construction is related to both Bousfield localization and Carlsson's derived completion.

1. Introduction

Quillen's classical plus-construction takes a pointed space X, with perfect normal subgroup P of $\pi_1(X)$, and produces a homology isomorphism $X \to X^+$ that induces the quotient map $\pi_1(X) \to \pi_1(X)/P$ on fundamental groups. The construction is typically introduced by attaching 2-cells and 3-cells via basic obstruction theory. The original interest in this construction is Quillen's theorem relating it to homotopy group completion: when applied to the classifying space of the infinite general linear group of a ring R, the result is a space whose homotopy groups are the higher algebraic K-groups of R.

This construction was later rephrased by Pirashvili [1] in the form of a plus-construction $G \to G^+$ for a simplicial group with a perfect normal subgroup of π_0. Using this form, he generalized the construction to Lie algebras. Livernet [2] later generalized the construction to algebras over a rational operad of chain complexes.

In this paper we define a plus-construction in stable homotopy theory. The approach is based on Quillen homology, generalizing topological André–Quillen homology in the commutative case and a structure related to topological Hochschild homology in the associative case.

Theorem 1. *Let $\mathcal{O}$ be a connective operad in symmetric spectra, and A a connective $\mathcal{O}$-algebra with an augmentation $A \to \mathcal{O}(0)$. Suppose J is a perfect $\pi_0(\mathcal{O})$-submodule of $\ker(\pi_n(A) \to \pi_n(\mathcal{O}(0)))$. Then there exists an n-connected map $A \to A^+$ of augmented $\mathcal{O}$-algebras inducing the quotient $\pi_n(A) \to \pi_n(A)/J$ such that the induced map on Quillen homology is an equivalence.*

In Section 2 we recall the definition of the (derived) Quillen homology object. In Section 3 we indicate how this can be computed as a simplicial object, and obtain the relevant obstruction theory in Section 4 to define a plus construction.

The constructions of this paper are carried out in a "based", or augmented, context. An unbased version would require a more delicate investigation of the homotopy of coproducts in $\mathcal{O}$-algebras and universal enveloping operads.

This is related to notions of Bousfield localization and completion. In Sections 6 and 7 we show that in the case where the augmentation ideal of an algebra A is perfect, the map $A \to A^+$ can also be identified as a Bousfield localization, and both notions are equivalent to the derived completion of Carlsson [3].

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1.1. Examples

We now indicate some examples in the case of associative and commutative algebras.

Example 2. The plus-construction projects off summands. If S and T are R-algebras and we have an augmentation factoring as $S \times T \to S \to R$, then the plus-construction corresponding to the perfect (i.e. idempotent) ideal $0 \times \pi_0(T) \subset \pi_0(S \times T)$ is equivalent to S.

This version, in some sense, provides the geometric description of the plus-construction for well-behaved rings: a map of pointed objects inducing an isomorphism on cotangent complexes at the basepoint locally models the inclusion of a component.

Several examples of the plus-construction are connected to known constructions on spaces or groups. The functor $R \wedge (-)$ takes smash products of spaces to smash products of R-modules, so given a topological or simplicial group G we can form the group ring spectrum

$$R[G] = R \wedge G_+.$$

This is an associative augmented R-algebra, and the Quillen homology is the desuspension of the fiber of the map

$$B(R, R[G], R) \simeq R \underset{R[G]}{\wedge} R \to R.$$

However, this fiber is naturally equivalent to the reduced homology object $R \wedge BG$. In particular, the Quillen homology groups of $R[G]$ are shifts of the reduced R-homology groups of BG.

Suppose $\pi_0 G$ has a normal subgroup P such that $\pi_0 R \otimes P_{ab} = 0$. Then the image of P is zero in the zeroth Quillen homology group $H_1(G, \pi_0 R)$, and $R[G]$ has a plus-construction annihilating elements of the form $(g - 1)$ for $g \in P$.

Example 3. If the group P is itself perfect, then the resulting associative ring spectrum can be taken to be $R[\Omega(BG^+)]$ for some strictly associative model of the loop space, such as the Moore loop space.

Example 4. Suppose $K \subset \mathrm{GL}_n(\mathbb{Z}_p)$ is a finite prime-to-p group acting on the abelian group $(\mathbb{Z}/(p^\infty))^n$. We can form the semidirect product

$$G = (\mathbb{Z}/(p^\infty))^n \rtimes K.$$

If p is nilpotent in $\pi_0 R$, then $R[G]$ has trivial zeroth Quillen homology, and so there is a plus-construction. A model for $R[G]^+$ is given by the group ring $R[\Omega(BG^\wedge_p)]$ of the associated p-compact group [4].

Example 5. Let G be a connected Lie group and G^δ the underlying discrete group. If $n = 0$ in $\pi_0 R$ for some $n > 0$, then $R[G^\delta]$ has trivial zeroth Quillen homology. There is an associated map $R[G^\delta] \to R[G]$, and this map being a plus construction for any such R is equivalent to a conjecture of Milnor on the relationship between the homology of G and G^δ [5].

If the Lie algebra of G is semisimple, then the group ring $R[G^\delta]$ always has trivial zeroth Quillen homology, and so we can form a plus-construction in general. Suppose $\pi_0 R$ is $\mathbb{Z}$. The second homology group of G^δ, which becomes isomorphic to the first homotopy group of $R[G^\delta]^+$, has as a quotient a positive eigenspace within the algebraic K-group $K_2(\mathbb{C})$ [6].

Example 6. Previous work [7] exhibits examples of derived completions in the case of representation rings of nilpotent groups, where a smaller ring of characters gives rise to a perfect ideal in the mod-p representation ring.

Notation and conventions. Throughout this paper we fix a base symmetric ring spectrum R, connective and commutative, that forms the ground ring for all smash products. Except where explicitly stated otherwise, all constructions are considered as happening in the derived sense.

2. Operads and Quillen homology

We begin by recalling work of Harper on operads in symmetric spectra [8], and give a short review of Quillen homology. Let $\mathcal{S}$ denote the category of symmetric spectra, and $\mathcal{S}^\Sigma$ the category of symmetric sequences in symmetric spectra, with composition product $\circ$.

Definition 7 (*[8, 3.3]*). An operad $\mathcal{O}$ in symmetric spectra is a symmetric sequence $\{\mathcal{O}(n)\} \in \mathcal{S}^\Sigma$ with the structure of a monoid under the composition product.

Theorem 8 (*[8, 1.3, 1.4]*). *Suppose $\mathcal{O}$ is an operad in symmetric spectra and let* $\mathrm{Alg}_\mathcal{O}$ *be the category of $\mathcal{O}$-algebras.*

- $\mathrm{Alg}_\mathcal{O}$ *has a natural model category structure where the forgetful functor to symmetric spectra (with the positive stable model structure) creates fibrations and weak equivalences.*

- *If $f: \mathcal{O} \to \mathcal{O}'$ is a map of operads, then there is a Quillen adjoint pair*

$$\mathrm{Alg}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Alg}_{\mathcal{O}'}$$

with f^* the restriction functor.
- *If such a map f is an objectwise stable equivalence, then the adjunction is a Quillen equivalence.*

Let $\mathcal{O}$ be an operad in R-modules. We begin by defining the following sequence of operads constructed from $\mathcal{O}$.

Definition 9. The nonunital $\mathcal{O}$-algebra operad $\mathcal{O}^+ \subset \mathcal{O}$ has $\mathcal{O}^+(k) = \mathcal{O}(k)$ for all $k > 0$ and $\mathcal{O}^+(0) = *$.

Definition 10. The $\mathcal{O}$-module operad $\mathcal{M}_{\mathcal{O}}(1) \subset \mathcal{O}$ has $\mathcal{M}_{\mathcal{O}}(1) = \mathcal{O}(1)$ and $\mathcal{M}_{\mathcal{O}}(k) = *$ if $k \neq 1$.

The symmetric spectrum $\mathcal{O}(1)$ is an associative R-algebra, and an algebra over $\mathcal{M}_{\mathcal{O}}$ is precisely a module over $\mathcal{O}(1)$.

Remark 11. If Com is the commutative operad with $\mathrm{Com}(k) = R$, the corresponding notions are of nonunital commutative algebras and R-modules. We drop the operad from the notation and simply write $\mathcal{M}$ for the category of R-modules.

The operad maps $\mathcal{M}_{\mathcal{O}} \to \mathcal{O}^+ \to \mathcal{O}$ induce "forgetful" functors $\mathbb{U}$ and adjoint "free", or pushforward, functors. We write $\mathbb{P}^+(M)$ and $\mathbb{P}(M)$ for the free nonunital $\mathcal{O}$-algebra and free $\mathcal{O}$-algebra on an $\mathcal{O}(1)$-module M.

If $\mathcal{O}$ is an operad, the category of augmented $\mathcal{O}$-algebras is the "over" category of $\mathcal{O}$-algebras X equipped with a map $\epsilon: X \to \mathcal{O}(0)$ of $\mathcal{O}$-algebras. The augmentation ideal $I(X)$ is the fiber of this augmentation.

The functor $\mathbb{P}$ naturally takes values in augmented $\mathcal{O}$-algebras with I as a right adjoint. As fibrations of $\mathcal{O}$-algebras are determined on the underlying objects, there is a well-defined right derived functor RI from augmented $\mathcal{O}$-algebras to nonunital $\mathcal{O}$-algebras. This is a Quillen equivalence.

There is a projection map $\pi: \mathcal{O}^+ \to \mathcal{M}_{\mathcal{O}}$. We define Q to be the "indecomposables" functor π_*, and E^+ the functor π^* inducing the trivial action of $\mathcal{O}(k)$ for $k > 1$.

The square-zero, or Eilenberg–Maclane, object associated with an $\mathcal{O}(1)$-module M is the augmented $\mathcal{O}$-algebra

$$E(M) = \mathcal{O}(0) \vee E^+(M).$$

Definition 12. The Quillen homology object of an augmented $\mathcal{O}$-algebra X is

$$QH(X) = LQ \circ RI(X).$$

This is an invariant of the homotopy type of the operad $\mathcal{O}$. We have an adjunction that descends to an adjunction on homotopy categories:

$$[X, E(M)]_{\mathcal{O}/\mathcal{O}(0)} \cong [QH(X), M]_{\mathcal{O}(1)}.$$

Definition 13. The wreath product algebra $\mathrm{Wr}(k, \mathcal{O})$ associated with $\mathcal{O}$ is the twisted group ring

$$\mathcal{O}(1)^{\wedge k} \wedge (\Sigma_k)_+,$$

where the multiplication map is given by

$$(x \wedge \sigma) \wedge (y \wedge \tau) \mapsto (x\sigma(y)) \wedge (\sigma\tau).$$

The spectra $\mathcal{O}(k)$ are $\mathcal{O}(1)$–$\mathrm{Wr}(k, \mathcal{O})$-bimodules. We note that there is a straightforward formula for the free $\mathcal{O}^+$-algebra.

Lemma 14. *For an $\mathcal{O}(1)$-module X, the free object on X is the $\mathcal{O}^+$-algebra*

$$\mathbb{P}^+ X = \bigvee_{k>0} \mathcal{O}(k) \underset{\mathrm{Wr}(k,\mathcal{O})}{\wedge} X^{\wedge k}.$$

If $\mathcal{O}(k)$ is a cofibrant $\mathcal{O}(1)$ –$\mathrm{Wr}(k, \mathcal{O})$-bimodule and X is cofibrant, then these smash products are derived smash products, and hence

$$L\mathbb{P}^+ X \simeq \bigvee_{k>0} \mathcal{O}(k) \overset{L}{\underset{\mathrm{Wr}(k,\mathcal{O})}{\wedge}} X^{\wedge^L k}.$$

Proof. The formula for the free $\mathcal{O}$-algebra is standard: it has the requisite universal property.

In order to show that the smash product is cofibrant, we note that

$$\mathcal{O}(k) \underset{\mathrm{Wr}(k,\mathcal{O})}{\wedge} X^{\wedge k} = \left[\mathcal{O}(k) \underset{\mathcal{O}(1)^{\wedge k}}{\wedge} X^{\wedge k} \right] / \Sigma_k.$$

The cofibrancy of X in the positive stable model structure ensures that $X^{\wedge k}$ has a free Σ_k-action. □

We relegate a proof of the following technical detail to Section 8.

Lemma 15. *There exists an equivalence of operads $\mathcal{P} \to \mathcal{O}^+$ such that $\mathcal{P}(k)$ is a cofibrant $\mathcal{P}(1)$ –$\mathrm{Wr}(k, \mathcal{P})$-bimodule for all $k > 1$.*

3. Bar constructions

In this section, we outline methods for computing obstruction groups for algebras over operads. The basic theorems required for these are based on the results of Harper [9], but also follow partially from results of Basterra [10] in the commutative case and Dugger and Shipley [11] in the associative case.

Let A be a cofibrant augmented $\mathcal{O}$-algebra, with $I(A)$ the associated augmentation ideal, and recall that $\mathcal{O}^+ \subset \mathcal{O}$ is the nonunital suboperad acting on $I(A)$. We may by Lemma 15 replace $\mathcal{O}^+$ with an operad $\mathcal{P}$ that is levelwise a cofibrant bimodule.

Theorem 16. *The derived Quillen homology object of A is equivalent to the geometric realization of the two-sided bar construction*

$$B(id, \mathbb{P}^+_{\mathcal{P}}, I(A)).$$

Proof. Harper shows in [9, 4.10] that it suffices to show that the simplicial bar construction $B(\mathbb{P}^+_{\mathcal{P}}, \mathbb{P}^+_{\mathcal{P}}, I(A))$ is a levelwise cofibrant $\mathcal{P}$-module. However, this follows from Lemma 14. □

The free $\mathcal{O}^+$-algebra on a cofibrant object is the left derived free $\mathcal{O}^+$-algebra. Hence, computing the homotopy spectral sequence of this simplicial spectrum gives us the following.

Corollary 17. *There exists a spectral sequence with E_1-term*

$$E_1^{p,q} = \pi_p((L\mathbb{P}^+)^{(q)} I(A)) \Rightarrow \pi_{p+q} QH(A),$$

with differentials induced by the monad structure of the left derived functor $L\mathbb{P}^+$.

Remark 18. The Quillen homology object, or linearization of an object, is the first layer in the Goodwillie tower for the forgetful functor from augmented $\mathcal{O}$-algebras to R-modules (the zeroth layer being $\mathcal{O}(0)$), and linear functors are roughly right $\mathcal{O}(1)$-modules. The category of symmetric k-multilinear functors is then equivalent to the category of right $\mathrm{Wr}(k, \mathcal{O})$-modules, and the higher layers in the Goodwillie tower identified with the spectra $\mathcal{O}(k)$. One can check this by evaluating the higher derivatives on free objects $\mathbb{P}(\mathcal{O}(1) \wedge S^n)$. (The author should remark that this is well-known to the experts, but does not know of a reference in the literature.)

4. The Hurewicz theorem

We now move on to calculations. From this point forward, by convention all objects are implicitly replaced by cofibrant or fibrant models so that appropriate derived functors are computed. In addition, we follow the convention that the homotopy groups of a symmetric spectrum X are not defined as a colimit of homotopy groups of the individual spaces, but as the group $[S^n, X]$ of maps in the homotopy category (or the classical homotopy groups of an appropriate Ω-spectrum replacement).

We apply the spectral sequence of Section 3 to obtain exact sequences of obstructions. Given a connective augmented $\mathcal{O}$-algebra A, let

$$P_N A \simeq \mathcal{O}(0) \vee I(A)[0 \ldots N]$$

be the Nth Postnikov section of A as an augmented $\mathcal{O}$-algebra. It has a unique $\mathcal{O}$-algebra structure (up to homotopy) making the map $A \to P_N A$ a map of augmented $\mathcal{O}$-algebras.

We write O for the operad $\pi_0(\mathcal{O}^+)$ and T for the O-algebra $\pi_0(I(A))$.

We note that the $\mathcal{O}^+$-algebra structure on $I(A)$ makes the groups $\pi_{N+1} I(A)$ into modules over this algebra in the sense that there are maps

$$O(k+1) \otimes T^{\otimes k} \otimes \pi_{N+1} I(A) \to \pi_{N+1} I(A)$$

that satisfy associativity and commutativity with respect to the operad composition.

Definition 19. For any module M over T, the decomposable submodule DM is the T-submodule which is the image of the map

$$\bigoplus_{k>0} O(k+1) \otimes T^{\otimes k} \otimes M \to M.$$

Proposition 20. *Suppose $A \to B$ is a map with first nonvanishing relative homotopy group $\pi_N(B, A) = J$ for some $N \geq 1$. Then the first nonvanishing relative homotopy group of the map $QH(A) \to QH(B)$ is*

$$\pi_N(QH(B), QH(A)) = J/DJ.$$

Proof. This follows by considering the map of bar complexes

$$B(id, \mathbb{P}^+, I(A)) \to B(id, \mathbb{P}^+, I(B))$$

inducing a map of Quillen homology spectral sequences. Taking homotopy fibers levelwise gives a bar construction whose realization is a homotopy fiber of the map $QH(A) \to QH(B)$.

A straightforward analysis of this levelwise fiber implies that π_N of the fiber of $\mathbb{P}^+I(A) \to \mathbb{P}^+I(B)$ accepts a surjective map from

$$X = \bigoplus_{k>0} O(k+1) \otimes_{\pi_0 \mathrm{Wr}(k) \otimes O(1)} T^{\otimes k} \otimes J.$$

(This map is an isomorphism if $N > 0$. If $N = 0$, this factors through the tensor product over $\pi_0\mathrm{Wr}(k+1)$.)

Therefore, the terms $E_1^{p,q}$ of the homotopy spectral sequence for the fiber of Quillen homology are zero unless $p \geq 0$, $q \geq N$, and take the following form:

$$\begin{array}{|ccc}
? & ? & ? \\
\mathrm{J} & \mathrm{X/K} & ? \\
0 & 0 & 0 \\
\vdots & & \\
\hline
\end{array}$$

The d_1-differential $X/K \to J$ is induced by the operad action and has image precisely consisting of the decomposable elements, proving that the desired exact sequence exists. □

Corollary 21. *The natural map*

$$\pi_i QH(A) \to \pi_i QH(P_N A)$$

is an isomorphism for $i < N$, and there exists a natural exact sequence

$$\pi_N I(A)/D\pi_N I(A) \to \pi_N QH(A) \to \pi_N QH(P_N A) \to 0.$$

Remark 22. For $N > 0$, the term in the spectral sequence in position $E_2^{N,1} = E_\infty^{N,1}$ is a module of *indecomposable relations*, i.e. operations on J that map to zero modulo those relations that can be deduced from operad composition.

We obtain a version of Whitehead's theorem.

Proposition 23. *If $\pi_0 I(A) = 0$ and $A \to B$ is a 0-connected map that is an equivalence on Quillen homology, then it is a weak equivalence.*

Proof. Applying Proposition 20, the first nonvanishing homotopy group J of the fiber of $A \to B$ would coincide with the submodule DJ of decomposables. Since $\pi_0 I(A) = \pi_0 I(B) = 0$, the decomposable submodule DJ is zero. □

5. The plus-construction

In this section, we obtain a plus-construction via the obstruction theory of Section 4.

Definition 24. Suppose that O is an operad in $\pi_0 R$-modules, T an O-algebra, and M a T-module. The module M is *perfect* if it coincides with the submodule DM of decomposable elements, i.e. the action map

$$\bigoplus_{k>0} O(k+1) \otimes T^{\otimes k} \otimes M \to M$$

is surjective.

Theorem 25. *Suppose $J \subset \pi_N I(A)$ is a perfect T-submodule. Then there exists an N-connected map $A \to A^+$ of augmented algebras inducing an isomorphism on Quillen homology and inducing the map $\pi_N A \to \pi_N A/J$ on homotopy.*

Proof. We follow a proof along the same lines as the ordinary plus construction.

Pick a set of T-module generators $\{e_\alpha\}$ for J, and let

$$X = \vee_\alpha \Sigma^N \mathcal{O}(1) \to I(A)$$

be the $\mathcal{O}(1)$-module map that takes the unit of each summand to the corresponding generator in J. We attach 1-cells as $\mathcal{O}$-algebras by forming the $\mathcal{O}$-algebra homotopy pushout

$$\begin{tikzcd} \mathbb{P}X \arrow[r] \arrow[d] & A \arrow[d] \\ \mathcal{O}(0) \arrow[r] & A'. \end{tikzcd}$$

We have $\pi_N A' = \pi_N A/J$. On Quillen homology, there is a homotopy pushout of $\mathcal{O}(1)$-modules

$$\begin{tikzcd} X \arrow[r] \arrow[d] & QH(A) \arrow[d] \\ * \arrow[r] & QH(A'). \end{tikzcd}$$

By Proposition 20, the map $A \to A'$ is N-connected, $\pi_N QH(P_N A) \to \pi_N QH(P_N A')$ is an isomorphism, and so since X is free there is a splitting

$$QH(A') \simeq QH(A) \vee \Sigma X$$

of $\mathcal{O}(1)$-modules.

Applying Corollary 21 to the maps $A \to P_N A$ and $A' \to P_N A'$, we obtain the following diagram of exact sequences:

$$\begin{tikzcd} \pi_{N+1} I(A) \arrow[r] \arrow[d] & \pi_{N+1} QH(A) \arrow[r] \arrow[d] & \pi_{N+1} QH(P_N A) \arrow[r] \arrow[d] & 0 \\ \pi_{N+1} I(A') \arrow[r] & \pi_{N+1} QH(A') \arrow[r] \arrow[d] & \pi_{N+1} QH(P_N A') \arrow[r] \arrow[d] & 0 \\ & \pi_N(X) \arrow[r] \arrow[d] & 0 & \\ & 0 & & \end{tikzcd}$$

By a standard diagram chase, we find that there is a map $\Sigma X \to I(A')$ of $\mathcal{O}(1)$-modules such that the composite

$$\Sigma X \to I(A') \to QH(A') \to \Sigma X$$

is an equivalence.

We then attach 2-cells by forming the $\mathcal{O}$-algebra homotopy pushout

$$\begin{tikzcd} \mathbb{P}(\Sigma X) \arrow[r] \arrow[d] & A' \arrow[d] \\ \mathcal{O}(0) \arrow[r] & A^+. \end{tikzcd}$$

The map $A' \to A^+$ is $(N+1)$-connected, and on Quillen homology there is a homotopy pushout of $\mathcal{O}(1)$-modules

$$\begin{tikzcd} \Sigma X \arrow[r] \arrow[d] & QH(A') \arrow[d] \\ * \arrow[r] & QH(A^+). \end{tikzcd}$$

Therefore the map $A \to A^+$ is N-connected and $\pi_N A^+ = \pi_N A/J$. The map $\Sigma X \to QH(A') \simeq QH(A) \vee \Sigma X$ reduces to the identity on ΣX, and so the composite map $QH(A) \to QH(A') \to QH(A^+)$ is an equivalence as desired. $\square$

Corollary 26. *If $\pi_0 I(A)$ is perfect, then any two plus-constructions A_1^+, A_2^+ killing $\pi_0 I(A)$ are weakly equivalent under A.*

Proof. Form the $\mathcal{O}$-algebra homotopy pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1^+ \arrow[d] \\ A_2^+ \arrow[r] & A_3^+. \end{tikzcd}$$

On Quillen homology, all the maps in this diagram are equivalences. The objects A_i^+ are all 0-connected, having $\pi_0 I(A_i^+) = 0$ by definition. Therefore, by Proposition 23 the two maps to A_3^+ are equivalences. □

6. Bousfield localization

We now specialize the constructions of Section 5 to the case where the operad $\mathcal{O}$ is either the commutative operad or the associative operad. In particular, $\mathcal{O}(0) = R$.

Assume A is a connective augmented $\mathcal{O}$-algebra with $\pi_0 I(A)$ satisfying $I^2 = I$, so that in particular the zeroth Quillen homology group vanishes. Let E be a cofibrant replacement of R viewed as an A-algebra via the augmentation; our goal is to examine the E-localization functor.

Lemma 27. *The natural map $E \wedge_A E \to E \wedge_{A^+} E$ is a weak equivalence.*

Proof. We carry this proof out in the associative and commutative cases separately.

If the operad $\mathcal{O}$ is the commutative operad, we note that there is a homotopy pushout diagram of augmented $\mathcal{O}$-algebras

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ E \arrow[r] & E \wedge_A E, \end{tikzcd}$$

and so the Quillen homology object of $E \wedge_A E$ is the suspension of the Quillen homology object of E (and similarly for A^+). Therefore, the map $E \wedge_A E \to E \wedge_{A^+} E$ is an equivalence on Quillen homology between objects whose augmentation ideals have vanishing π_0, and hence by Proposition 23 it is a weak equivalence.

If the operad $\mathcal{O}$ is instead the associative operad, the Quillen homology object of A is the fiber of the multiplication map

$$E \underset{A}{\wedge} E \to E.$$

(This is a "standard" relationship between topological Hochschild homology and Quillen homology for associative algebras.) Therefore, the equivalence on Quillen homology implies that map

$$E \underset{A}{\wedge} E \to E \underset{A^+}{\wedge} E$$

is a weak equivalence. □

We have a Bousfield localization functor

$$X \mapsto L_E X$$

on the homotopy category of A-modules. A map $X \to Y$ is an E-equivalence (and induces an equivalence on localizations) if and only if $E \wedge_A X \to E \wedge_A Y$ is a weak equivalence, and an object Z is E-local if the functor $[-, Z]_A$ takes E-equivalences to isomorphisms.

Theorem 28. *There is a natural equivalence*

$$L_E A \simeq A^+$$

from the Bousfield localization of A to the plus-construction.

Proof. We first show that A^+ is E-local. We express the object A^+ as the homotopy limit of its Postnikov stages $A^+[0 \ldots N]$, which are naturally A^+-modules. We have $A^+[0] \simeq H\pi_0 E$, and for any N we have homotopy fiber sequences of A^+-modules

$$\Sigma^N H\pi_N A^+ \to A^+[0 \ldots N] \to A^+[0 \ldots N-1].$$

The object $H\pi_N A^+$, as an Eilenberg–Maclane spectrum and an A^+-module, naturally inherits the structure of a module over $H\pi_0 A^+ = H\pi_0 E$. Therefore, it can be given the structure of an E-module, and so the standard adjunction

$$[X, H\pi_N A^+]_A \cong [E \underset{A}{\wedge} X, H\pi_N A^+]_E$$

shows that $H\pi_N A^+$ is E-local.

Local objects are closed under taking homotopy fibers and homotopy limits, and hence $A^+ \simeq \operatorname{holim} A^+[0\ldots N]$ is E-local.

We now show that the map $A \to A^+$ is an E-equivalence, or equivalently that the map $E \to E \wedge_A A^+$ is a weak equivalence. By Lemma 27, we find that the map

$$(E \underset{A}{\wedge} A^+) \underset{A^+}{\wedge} E \to (E \underset{A^+}{\wedge} A^+) \underset{A^+}{\wedge} E$$

is a weak equivalence.

Let F be the fiber of the map $E \wedge_A A^+ \to E \wedge_{A^+} A^+$ of connective right A^+-modules. Then $F \wedge_{A^+} E$ is contractible. However, the map $\pi_0 A^+ \to \pi_0 E$ is an isomorphism, and so the Künneth spectral sequence

$$\operatorname{Tor}^{\pi_* A^+}_{**}(\pi_* F, \pi_* E) \Rightarrow \pi_*(F \underset{A^+}{\wedge} E) = 0$$

implies that F is contractible. Hence the map $A \to A^+$ is an E-equivalence as desired. □

Remark 29. Minasian has given a spectral sequence computing topological Hochschild homology from topological André–Quillen homology [12, Corollary 2.7]. This shows directly that a map of commutative symmetric ring spectra inducing an equivalence on topological André–Quillen homology induces one on topological Hochschild homology.

Remark 30. The results of the theorem should hold true in categories of algebras over the little n-cubes operad. The Quillen homology functor on augmented E_n-algebras should factor through Quillen homology for A_∞-algebras. However, the author is not aware of a sufficient reference in the literature for "iterating the bar construction" to rigorously justify this argument.

7. Derived completion

Carlsson [3] defines a notion of "derived completion" of A with respect to E (the E-nilpotent completion of Bousfield) as the totalization of a cosimplicial object

$$\mathcal{T}^{\cdot}_A(A; E)^p = \left\{ E^{\wedge_A (p+1)} = E \underset{A}{\wedge} E \underset{A}{\wedge} \cdots \underset{A}{\wedge} E \right\}.$$

Here the coface maps are induced by the unit $A \to E$ and the codegeneracy maps are induced by the multiplication $E \wedge_A E \to E$. This is isomorphic to the totalization of the standard cobar construction associated with the "Hopf algebroid" $(E, E \wedge_A E)$ in spectra. The spectral sequence for the homotopy of the totalization is a generalized Adams–Novikov spectral sequence abutting to $\pi_{t-s} A$ based on E-homology. When $\pi_*(E \wedge_A E)$ is flat over E_*, this spectral sequence has an identifiable E_2-term

$$\operatorname{Ext}^{s,t}_{(\pi_* E, \pi_*(E \underset{A}{\wedge} E))}(\pi_* E, \pi_* E)$$

as in [13]. (In the particular case where A is the group ring $R[G]$ of a topological group G, the derived completion spectral sequence is the Eilenberg–Moore spectral sequence abutting to the R-homology of ΩBG.)

We continue the assumptions of the previous section: $\mathcal{O}$ is either the commutative or associative operad over R, and A is a connective $\mathcal{O}$-algebra with the augmentation ideal $I = \pi_0 I(A)$ satisfying $I = I^2$. Again, let E be a cofibrant replacement of R regarded as an A-algebra.

By Lemma 27, the map $E \wedge_A E \to E \wedge_{A^+} E$ is a weak equivalence, and hence induces an isomorphism of Adams–Novikov spectral sequences. However, the map $A^+ \to E$ is 1-connected by construction. The augmentation $E \wedge_A E \to E$ is then 2-connected, and so the Adams–Novikov spectral sequence has a vanishing line because the homotopy groups in the reduced cobar complex $E_1^{s,t}$ vanish for $(t-2s) < 0$. Hence this spectral sequence converges strongly. This implies that A^+ is E-local as in Theorem 28 and that the E-Bousfield localization and E-nilpotent completion of A^+ coincide [13, Theorem 2.4].

As a result, the natural diagram of algebras and their derived completions

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A^+ \arrow[d, "\sim"] \\ A^\wedge_E \arrow[r, "\sim"] & (A^+)^\wedge_E \end{tikzcd}$$

shows that A^+ and the derived completion $A^\wedge_E$ are weakly equivalent.

8. Cofibrant replacement of operads

In this section we sketch a proof of Lemma 15.

The category of positively graded symmetric sequences in R-modules admits a levelwise model structure where the forgetful functors $Y \mapsto Y(n)$ to $R[\Sigma_n]$-modules (in the positive stable model structure) create cofibrations, fibrations, and weak equivalences. This is a (non-symmetric) monoidal closed category under the composition product $\circ$:

$$(A \circ B)(n) = \bigvee_{\sum kj_k = n} \left(A(\Sigma j_k) \wedge \bigwedge_k B(k)^{\wedge j_k} \right) \underset{R\left[\prod \Sigma_{j_k} \wr \Sigma_k\right]}{\wedge} R[\Sigma_n]$$

Cofibrant objects are levelwise cofibrant R-modules, and the composition product preserves the property of being a relative cell inclusion in each variable. We would like to call this a monoidal model category, but this conflicts with the definition of [14] that requires the underlying category to be symmetric monoidal.

We claim that a general algebra $\mathcal{P}$ in this model category can be replaced by a new algebra $\mathcal{P}' \to \mathcal{P}$ such that $\mathcal{P}'(k)$ is a cofibrant $\mathcal{P}'(1) - \mathrm{Wr}(k, \mathcal{P}'(1))$-bimodule for $k > 1$.

Let $\mathcal{P}'(1)$ be a cofibrant replacement of $\mathcal{P}(1)$ as an associative algebra, and let $\mathcal{N}$ be $\mathcal{P}'(1)$ concentrated in degree 1. The construction of Shipley and Schwede [14, Lemma 6.2] allows one to factor the map $\mathcal{N} \to \mathcal{P}$ through an operad $\mathcal{P}'$ constructed via a possibly transfinite sequence of pushouts in $\mathcal{S}p^{\Sigma}$ of the form

$$\begin{tikzcd}
\mathcal{N} \circ X \circ \mathcal{N} \arrow[r] \arrow[d] & \mathcal{N} \circ Y \circ \mathcal{N} \arrow[d] \\
\mathcal{P}_\alpha \arrow[r] & \mathcal{P}_{\alpha+1},
\end{tikzcd}$$

for $X \to Y$ a cofibration of R-modules and $\mathcal{P}_\alpha$ an $\mathcal{N}$-bimodule. The resulting object $\mathcal{P}'$ would then be levelwise cofibrant, proving Lemma 15.

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